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Free field realization of D-brane in group manifold

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Abstract: We construct the boundary state for the D-brane in the SU(2) group manifold directly in terms of the group variables. We propose a matching condition for the left- and the right-moving sectors including the zero modes that describes a D-brane of the Neumann-type. The free field realization of the WZW model is used to obtain the boundary state subject to the matching condition. We show that the resulting state coincides with Cardy's state. The structure of the BRST cohomology is realized by imposing the invariance of the state under the Weyl group of the current algebra.

Keywords: Bosonic Strings, D-branes, Conformal Field Models in String Theory, Boundary Quantum Field Theory.

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1. Introduction

Various properties of D-branes in flat backgrounds are now well understood from the sigma model approach as well as from the exact conformal field theory (CFT) approach. In the former one, a D-brane is a submanifold of the target space on which the boundary of the worldsheet is lying. By construction, geometrical data such as the position and the shape of the brane are obvious. In the latter case, we have a boundary CFT and a D-brane is described by the boundary state. The geometrical data of branes are encoded in the boundary states and can be retrieved by acting the string coordinate on the states.

In the case of curved backgrounds, however, this correspondence between the geometry of D-branes and the boundary states is not so straightforward since we do not know the precise relation between the exact CFT and the target space. Even if we can construct a boundary state subject to an appropriate boundary condition, we are left with the problem to extract the geometrical information from the resulting state.

The D-branes in group manifolds have recently attracted much attention [1]–[11]. They give us an example of curved branes in the non-trivial backgrounds. The corresponding worldsheet theory is the Wess-Zumino-Witten (WZW) model and is exactly solvable. We can therefore take both approaches of the sigma model and the exact CFT to analyze the branes. A group manifold is a good example to study the geometrical interpretation of D-branes.[1]

[1]There are also several works studying the case of the Calabi-Yau manifold [18]–[21].

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In the WZW model on the group manifold G, we have the $G \times G$ symmetry of the left- and the right-translations. Correspondingly, we have the left- and the right-moving current, $J(z)$ and $\tilde{J}(\bar{z})$, respectively. The gluing condition that preserves the diagonal part of $G \times G$ takes the following form:

$$J + \tilde{J} = 0 \,. \tag{1.1}$$

This condition implies that the worldvolume of the D-brane coincides with a conjugacy class of G [3, 12]. In exact CFT, one can solve the same problem to obtain the boundary states satisfying the above condition [13, 14]. Since these two approaches treat the same object, we expect that it is possible to relate the boundary states with the conjugacy class of G. Although there is an argument [12, 6] based on the wave function of the string zero modes, a more direct correspondence including the oscillator modes is desirable.

In this paper, we study the above problem of the relation between the geometry of D-branes and the boundary states for the case of $G = \mathrm{SU}(2)$, focusing on the construction of boundary states in terms of group variables. We treat directly the left- and the right-moving group variables, $g_L(z)$ and $g_R(\bar{z})$, rather than the currents $J(z)$ and $\tilde{J}(\bar{z})$. By using the group variables, the geometrical meaning of the boundary state is manifest.

This paper is organized as follows. In the next section, we review some of the known results about the D-brane in the SU(2) group manifold. We also discuss the structure of the wave function for the boundary states found by Cardy [14]. In section 3, we propose the matching condition of the left- and the right-moving group variables corresponding to the D-brane wrapped around the conjugacy class. We then construct the boundary states which satisfy the matching condition using the free field realization [15, 16, 17]. We require invariance of the states under the Weyl group of the current algebra, and show that the resulting states coincide with Cardy's states. This fact clarifies the geometrical meaning of Cardy's states. The final section is devoted to discussions.

2. D-brane in the $\mathrm{SU}(2)$ group manifold

In this section, we briefly review some properties of the D-branes in the WZW model.

We first recall some basic facts about the functions on a group manifold. Let (π, V_π) be a unitary irreducible representation of a compact group G and take $\{|m\rangle\}$ to be an orthonormal basis of V_π. Then, the matrix element π_{mn} is written as

$$\pi_{mn}(g) = \langle m|\pi(g)\,|n\rangle\,, \quad g \in G\,, \tag{2.1}$$

or equivalently,

$$\pi(g)\,|n\rangle = |m\rangle\,\pi_{mn}(g)\,. \tag{2.2}$$

The set $\{\pi_{mn}\}$ form a basis of $L^2(G)$, the space of all square integrable functions on G. They are orthogonal to each other,[2]

$$\int_G dg\, \overline{\pi_{ij}(g)}\pi'_{kl}(g) = \begin{cases} 0 & \text{if } \pi \neq \pi' , \\ \frac{1}{\dim \pi}\delta_{ik}\delta_{jl} & \text{if } \pi = \pi', \end{cases} \tag{2.3}$$

which is known as Schur's orthogonality relation.

We denote the conjugacy class including an element t as $\mathcal{C}(t)$,

$$\mathcal{C}(t) = \left\{ gtg^{-1} ,\ g \in G \right\} . \tag{2.4}$$

A function $\psi(x)$ invariant under the adjoint action $x \to gxg^{-1}$ of G is called a class function, since it is constant along the conjugacy class. The character $\chi_\pi(g)$ of the representation π is defined as

$$\chi_\pi(g) = \operatorname*{Tr}_{V_\pi} \pi(g) = \sum_m \pi_{mm}(g) . \tag{2.5}$$

One can take χ_π as an orthonormal basis of the class functions. In fact,

$$\int_G dg\, \overline{\chi_\pi(g)}\chi'_\pi(g) = \delta_{\pi\pi'} , \tag{2.6}$$

which follows from (2.3).

The δ-function $\delta(g,g')$ on G is characterized by the equation

$$f(g) = \int_G dg'\, \delta(g,g')f(g') , \quad f \in L^2(G) . \tag{2.7}$$

We can write down the δ-function in terms of π_{mn},

$$\delta(g,g') = \sum_{\pi;m,n} (\dim \pi)\, \pi_{mn}(g)\overline{\pi_{mn}(g')} = \sum_\pi (\dim \pi)\, \chi_\pi(gg'^{-1}) , \tag{2.8}$$

where the sum is over all the unitary irreducible representations of G. Making a superposition of δ-functions over the conjugacy class $\mathcal{C}(t)$, we can also write the δ-function δ_t that concentrates on $\mathcal{C}(t)$,

$$\delta_t(g) = \int_G dg'\, \delta(g'tg'^{-1}, g) = \sum_\pi \chi_\pi(t)\overline{\chi_\pi(g)} . \tag{2.9}$$

From the left- and the right-translation on G, we obtain a natural action of $G\times G$ on $L^2(G)$,

$$\begin{aligned} (L_g\psi)(x) &= \psi(g^{-1}x) , \\ (R_g\psi)(x) &= \psi(xg) . \end{aligned} \tag{2.10}$$

[2]Here, $\overline{\pi}$ stands for the complex conjugate of π.

Taking $\psi = \overline{\pi}_{mn}$, we obtain

$$\begin{aligned}(L_g\overline{\pi}_{mn})(x) &= \overline{\pi}_{ln}(x)\pi_{lm}(g)\,, \\ (R_g\overline{\pi}_{mn})(x) &= \overline{\pi}_{ml}(x)\overline{\pi}_{ln}(g)\,,\end{aligned} \tag{2.11}$$

which means that $\overline{\pi}_{mn}$ transforms as $\pi \times \overline{\pi}$ of $G \times G$. $L^2(G)$ is therefore decomposed in the following form:

$$L^2(G) \cong \bigoplus_i V_{\pi_i} \otimes V_{\overline{\pi_i}}\,, \tag{2.12}$$

where the sum is over all the unitary irreducible representations of G.

The wave function of a particle on G is an element of $L^2(G)$. Hence, the Hilbert space of the quantum mechanics on G is identified with $L^2(G)$ and has the structure of (2.12). Using the basis given in (2.2), we can write a basis of the Hilbert space in the form of $|m\rangle \otimes \overline{|n\rangle}$. From the transformation property (2.11) of π_{mn}, we can identify $|m\rangle \otimes \overline{|n\rangle}$ with the wave function $\overline{\pi}_{mn}$,

$$\overline{\pi}_{mn} \longmapsto \frac{1}{\sqrt{\dim \pi}}\,|m\rangle \otimes \overline{|n\rangle}\,, \quad |m\rangle\,, |n\rangle \in V_\pi\,. \tag{2.13}$$

The factor $1/\sqrt{\dim \pi}$ is necessary to preserve the inner product of (2.3).

Let us consider the case of string theory. In the WZW model, we have two chiral currents J and $\tilde{J}$ corresponding to the left- and the right-translation on G:

$$\begin{aligned}J(z) &= \sum_n J_n^a T^a\, z^{-n-1} = -k\,\partial g\, g^{-1}\,, \\ \tilde{J}(\bar{z}) &= \sum_n \tilde{J}_n^a T^a\, \bar{z}^{-n-1} = k\, g^{-1}\bar{\partial} g\,.\end{aligned} \tag{2.14}$$

Here we parametrize the worldsheet by the coordinates $(z, \bar{z})$, and k is the level of the model. The T^a are generators of the Lie algebra of the group G.

In the presence of D-branes, the string worldsheet has boundaries and we have to impose an appropriate gluing condition for the currents. The most natural choice is the Neumann-type[3]

$$J_n^a + \tilde{J}_{-n}^a = 0\,, \tag{2.15}$$

which preserves the diagonal part of the $G \times G$ symmetry on the worldsheet. Since the energy-momentum tensor $T(z) = \sum_n L_n z^{-n-2}$ of the WZW model is a quadratic form of the current $J(z)$, the above condition assures that half of the conformal symmetry is also preserved:

$$L_n - \tilde{L}_{-n} = 0\,. \tag{2.16}$$

Hence, the gluing condition (2.15) is compatible with the conformal invariance.

[3]Here we take the closed-string point of view; the boundary is placed at $\mathrm{Re}\, z = 0$. In the open-string channel, we take the boundary at $\mathrm{Im}\, z = 0$ and the Neumann boundary condition turns out to be $J_n^a = \tilde{J}_{-n}^a$.

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As is shown in [3], the Neumann-type condition (2.15) implies that the boundary of the worldsheet is constrained within one of the conjugacy classes of G,

$$g(z,\bar{z})|_{z+\bar{z}=0} \in \mathcal{C}(t)\,. \tag{2.17}$$

In other words, the worldvolume of the D-brane coincides with the conjugacy classes. In order to understand this result, let us consider the wave function $\psi(x)$ for the string zero mode. As eq. (2.10) shows, $J + \tilde{J}$ is the generator of the adjoint action $x \to gxg^{-1}$. The gluing condition (2.15) therefore means that the wave function should obey the relation $\psi(gxg^{-1}) = \psi(x)$, i.e. the wave function is a class function. Among the class functions, we have the δ-function δ_t given in (2.9) that is concentrated on a specific conjugacy class $\mathcal{C}(t)$. If we take it as the wave function of string, the corresponding state is nothing but a D-brane wrapped around the conjugacy class $\mathcal{C}(t)$.

For $G = \mathrm{SU}(2) \cong S^3$, a generic conjugacy class is isomorphic to a 2-sphere S^2. In addition to this, there are two conjugacy classes, $\mathcal{C}(1)$ and $\mathcal{C}(-1)$, consisting of a point. We can parametrize the conjugacy classes by the element $t = e^{i\theta\sigma^3}$ of the maximal torus $\mathbb{T} \cong S^1$.[4] More precisely, what parametrizes the conjugacy class is $\mathbb{T}/W$, where W is the Weyl group of G. For $G = \mathrm{SU}(2)$, $W = \mathbb{Z}_2$ and t takes values in $S^1/\mathbb{Z}_2$. The parameter $\theta \in [0,\pi]$ coincides with the latitude in S^3, where $\theta = 0$ and π correspond to the north and the south pole of S^3, respectively. From the quantization of the Dirac flux on the worldvolume, only a finite number of classes are allowed [3, 12], namely $\theta = \pi\frac{2\alpha}{k}$, $\alpha = 0, 1/2, 1, \ldots, k/2$. We therefore have $k-1$ 2-branes, which correspond to the 'regular' conjugacy classes, and two 0-branes, which are located at the north and the south pole.

If we suppose that the worldvolume of the D-brane exactly coincides with the conjugacy class, the wave function of the string zero mode for the boundary state would be the δ-function (2.9). For the D-brane at $\theta = \pi\frac{2\alpha}{k}$, we obtain the wave function ψ^k_α as follows:

$$\psi^k_\alpha(\theta) \equiv \delta_{t=e^{i\pi\frac{2\alpha}{k}\sigma^3}}(e^{i\theta\sigma^3}) = \sum_{j=0,1/2,1,\ldots} \chi_j\left(\pi\frac{2\alpha}{k}\right)\overline{\chi_j(\theta)}\,. \tag{2.18}$$

Here, we write the su(2) character χ_j of the irreducible representation with spin j as a function of θ rather than the group element t,

$$\chi_j(\theta) = \operatorname*{Tr}_{V_j} e^{i\theta\sigma^3} = \frac{\sin((2j+1)\theta)}{\sin\theta}\,, \tag{2.19}$$

where V_j is the representation space of spin j.

We can analyze the branes in SU(2) from the point of view of the exact CFT. The WZW model for $G = \mathrm{SU}(2)$ with level k is described by a CFT with the current algebra $\widehat{\mathrm{su}}(2)_k$ and the diagonal modular invariant partition function. In this CFT,

[4] Our convention is $\sigma^1 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\sigma^2 = \left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$, $\sigma^3 = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$.

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there are a finite number of primary fields, which are labeled by the $\widehat{\mathrm{su}}(2)_k$ integrable highest weight $j = 0, 1/2, 1, \ldots, k/2$. The Hilbert space of the CFT therefore takes the form

$$\bigoplus_{j=0,1/2,1,\ldots,k/2} \widehat{V}_j \otimes \widehat{V}_{\bar{j}}\,, \tag{2.20}$$

where $\widehat{V}_j$ is the irreducible representation of $\widehat{\mathrm{su}}(2)_k$ with spin j and $\bar{j}$ is the complex conjugate of the su(2) representation j. The left-moving sector of string acts on $\widehat{V}_j$ while the right-moving one acts on $\widehat{V}_{\bar{j}}$. This structure of the Hilbert space is analogous to the spectrum (2.12) of a particle on G. Actually, the ground states of $\widehat{V}_j$ transform as the representation of spin j under the action of the su(2) subalgebra J_0^a. They represent the wave function of the zero modes of string and we denote them as V_j. Schematically, $\widehat{V}_j = \{J_{-n}^a\}V_j$.

For the worldsheet with boundaries, we have to impose the gluing condition (2.15) at the boundary. The boundary states subject to the gluing condition (2.15) have been obtained by Ishibashi [13] and are called the Ishibashi states. They are labeled by the spin j and we denote them as $|j\rangle\rangle_I$

$$|j\rangle\rangle_I = \sum_N |j, N\rangle \otimes \overline{|j, N\rangle}\,, \tag{2.21}$$

where $|j, N\rangle$ is an orthonormal basis of $\widehat{V}_j$ and $\overline{|j, N\rangle}$ is its complex conjugate. One can show that the Ishibashi states $|j\rangle\rangle_I$ satisfy the following equation

$${}_I\langle\langle j| q^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{24})} z^{\frac{1}{2}(J_0^3-\tilde{J}_0^3)} |j'\rangle\rangle_I = \delta_{jj'}\hat{\chi}_j(\tau, \nu)\,. \tag{2.22}$$

We write $q = e^{2\pi i\tau}$, $z = e^{2\pi i\nu}$, and the central charge c takes the value $c = \frac{3k}{k+2}$. $\hat{\chi}_j$ is the character of the irreducible representation $\widehat{V}_j$ of $\widehat{\mathrm{su}}(2)_k$,

$$\hat{\chi}_j(\tau, \nu) = \operatorname*{Tr}_{\hat{V}_j} q^{L_0-\frac{c}{24}} z^{J_0^3} = \frac{\vartheta_{2j+1,k+2} - \vartheta_{-2j-1,k+2}}{\vartheta_{1,2} - \vartheta_{-1,2}}(\tau, \nu)\,, \tag{2.23}$$

where the theta function $\vartheta_{a,b}$ is

$$\vartheta_{a,b}(\tau, \nu) = \sum_{m\in\mathbb{Z}+\frac{a}{2b}} \exp(2\pi i b(\tau m^2 + \nu m))\,. \tag{2.24}$$

A generic boundary state satisfying the gluing condition (2.15) is a linear combination of the Ishibashi states. In this sense, the Ishibashi state is the building block of the boundary states. However, we can not take an arbitrary combination of them because of the constraint from the modular invariance [14]. Suppose that we have two boundary conditions α and β satisfying the gluing condition (2.15), and denote the corresponding boundary state as $|\alpha\rangle\rangle$ and $|\beta\rangle\rangle$. We can calculate the annulus

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amplitude with the boundary conditions α and β in two different ways. From the closed string channel, the amplitude can be calculated as

$$\langle\!\langle \alpha | q^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{24})} z^{\frac{1}{2}(J_0^3-\tilde{J}_0^3)} | \beta \rangle\!\rangle\,. \tag{2.25}$$

From the open string channel, the above amplitude is the trace over the open string Hilbert space and counts the spectrum of the open string with the boundary conditions α and β. Since the gluing condition (2.15) preserves the $\widehat{\mathrm{su}}(2)_k$ symmetry, the spectrum is a direct sum of the irreducible representations of $\widehat{\mathrm{su}}(2)_k$. The annulus amplitude (2.25) should therefore be a sum of the $\widehat{\mathrm{su}}(2)_k$ characters with integer coefficients. This requirement for the annulus amplitude restricts the form of the boundary states. Cardy [14] solved this problem and obtained the allowed form of the states. We call them Cardy's states and denote them as $|\alpha\rangle\!\rangle_C$

$$|\alpha\rangle\!\rangle_C = \sum_j S_{\alpha j} \frac{1}{\sqrt{S_{0j}}} |j\rangle\!\rangle_I\,. \tag{2.26}$$

Here the sum is over all the integrable highest weights $j = 0, 1/2, 1, \ldots, k/2$. S_{ij} is the modular transformation matrix[5]

$$\hat{\chi}_i\left(-\frac{1}{\tau}\right) = \sum_j S_{ij}\hat{\chi}_j(\tau)\,. \tag{2.27}$$

For $\widehat{\mathrm{su}}(2)_k$, it takes the following form:

$$S_{ij} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(2i+1)(2j+1)}{k+2}\right)\,. \tag{2.28}$$

As is seen from the definition (2.26), Cardy's states $|\alpha\rangle\!\rangle_C$ are also labeled by the integrable highest weight $\alpha = 0, 1/2, 1, \ldots, k/2$. Hence we have $k+1$ boundary states, or $k+1$ branes, which are subject to the gluing condition (2.15) and satisfy the modular invariance.

The map (2.13) between the square integrable function and the state of quantum mechanical particle on G enables us to read off the wave function corresponding to the boundary state. For example, the wave function for the Ishibashi state (2.21) can be obtained as follows:

$$\begin{aligned} |j\rangle\!\rangle_I &= \sum_{-j\leq m\leq j} |j,m\rangle \otimes \overline{|j,m\rangle} + \text{states with oscillators} \\ &\longmapsto \sqrt{2j+1}\sum_{-j\leq m\leq j} \overline{\pi}_{mm} = \sqrt{2j+1}\,\overline{\chi}_j\,. \end{aligned} \tag{2.29}$$

Here $|j,m\rangle$ is the orthonormal basis of V_j and m is the weight $J_0^3 = m$. This equation shows that the Ishibashi state $|j\rangle\!\rangle_I$ corresponds to the su(2) character χ_j up

[5]Here, we set $\nu = 0$.

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to normalization. Using this fact, we can obtain the wave function $\widehat{\psi}^k_\alpha$ corresponding to Cardy's state $|\alpha\rangle\rangle_C$[6]

$$\begin{aligned}
|\alpha\rangle\rangle_C &= \sum_{j=0,1/2,1,\ldots,k/2} S_{\alpha j}\frac{1}{\sqrt{S_{0j}}}|j\rangle\rangle_I \\
&\longmapsto \widehat{\psi}^k_\alpha(\theta) \equiv \frac{S_{\alpha 0}}{\sqrt{S_{00}}}\sum_{j=0,1/2,1,\ldots,k/2}\sqrt{\frac{2j+1}{[2j+1]_q}}\chi_j\left(\pi\frac{2\alpha+1}{k+2}\right)\overline{\chi}_j(\theta)\,. \quad (2.30)
\end{aligned}$$

Here we have used the relations

$$\frac{S_{\alpha j}}{S_{\alpha 0}} = \chi_j\left(\pi\frac{2\alpha+1}{k+2}\right), \tag{2.31}$$

$$\frac{S_{0j}}{S_{00}} = \chi_j(0) = [2j+1]_q\,, \quad q = e^{i\frac{2\pi}{k+2}}\,, \tag{2.32}$$

and the q-number $[n]_q$ defined by

$$[n]_q = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}\,. \tag{2.33}$$

One can see that the wave function $\widehat{\psi}^k_\alpha$ for Cardy's state $|\alpha\rangle\rangle_C$ has almost the same form as the wave function ψ^k_α (2.18) obtained under the assumption that the world-volume of the D-brane exactly coincides with the conjugacy class. The differences between these two are: (1) $\widehat{\psi}^k_\alpha$ has the additional factor $\sqrt{\frac{2j+1}{[2j+1]_q}}$ (2) The sum over the spin j is restricted to $j \leq k/2$ for $\widehat{\psi}^k_\alpha$ (3) k and 2α is shifted to $k+2$ and $2\alpha+1$, respectively, in $\widehat{\psi}^k_\alpha$. Clearly, these differences disappear in the limit of $k\to\infty$ with $\frac{2\alpha}{k}$ fixed, since $[n]_q$ reduces to n as $q\to 1$. Therefore, we can regard the wave function ψ^k_α as the $k\to\infty$ (or the classical) limit of the wave function $\widehat{\psi}^k_\alpha$ for Cardy's state. For finite k, the D-brane wrapped around the conjugacy class is described by Cardy's state. We can consider that the difference between ψ^k_α and $\widehat{\psi}^k_\alpha$ for finite k is the quantum effect, which vanish in the classical limit $k\to\infty$.

Instead of the gluing condition (2.15), one can take a "twisted" form [2]

$$J^a_n + \omega(\tilde{J}^a_{-n}) = 0\,, \tag{2.34}$$

where ω is a Lie algebra automorphism. This includes the Dirichlet boundary condition in the case of the flat background, for which ω acts as a rotation of the right-moving coordinates. Since ω does not change the energy-momentum tensor, the conformal invariance is also preserved under the twisted gluing condition (2.34). The construction of the boundary states proceeds in the way similar to the ordinary gluing condition [2].

[6]The wave function for Cardy's states is also discussed in ref. [6].

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For the WZW model, a natural choice for ω is an automorphism of the horizontal subalgebra. For such ω, the worldvolume corresponding to the gluing condition (2.34) is a "twisted" version of the conjugacy class [6]

$$\mathcal{C}^{\omega}(t) = \left\{ g t \omega(g)^{-1} ,\ g \in G \right\} . \tag{2.35}$$

This fact is easily understood in terms of the wave function, in the same way as the Neumann-type condition (2.15). Namely, with the gluing condition (2.34), one can show that the wave function $\psi(x)$ should satisfy $\psi(gx\omega(g)^{-1}) = \psi(x)$, which is obviously related with the twisted conjugacy class (2.35).

If we take ω as an inner automorphism $\omega(\lambda) = \omega\lambda\omega^{-1}$, C^{ω} is isomorphic to the ordinary conjugacy class (2.4)

$$C^{\omega}(t) = \left\{ g t (\omega g \omega^{-1})^{-1} ,\ g \in G \right\} = \left\{ g (t\omega) g^{-1} \cdot \omega^{-1} ,\ g \in G \right\} = C(t\omega)\omega^{-1} . \tag{2.36}$$

On the other hand, for an outer automorphism, C^{ω} has in general a topology different from the ordinary class [6]. One can regard this fact as a generalization of the well-known behavior of D-branes on a torus under the T-duality transformation. As an example, we take a T-duality transformation $X_L(z)+X_R(\bar{z}) \to X_L(z)-X_R(\bar{z})$, which acts on the current $\tilde{J} = i\bar{\partial}X_R$ as an outer automorphism $\tilde{J} \to -\tilde{J}$. The dimension of the D-brane worldvolume changes by one under this transformation, which shows that an outer automorphism can change the topology, the dimension in this case, of the worldvolume.

3. Free field realization of the boundary condition

3.1 Matching condition

One of the characteristic features of conformal field theory in two dimensions is the existence of chiral and anti-chiral sectors commuting with each other. For a string coordinate X of toroidal compactification, which takes values in $S^1 \cong \mathbb{R}/\mathbb{Z}$, we have a decomposition

$$X(z, \bar{z}) = X_L(z) + X_R(\bar{z}) . \tag{3.1}$$

This follows from the equation of motion $\partial\bar{\partial}X = 0$. Since there is no real holomorphic function except for a constant, we can not decompose X into X_L and X_R within the real functions. This would be the case if we took the Minkowski metric, instead of the euclidean metric, on the worldsheet.

One can regard the boundary condition of a string as a matching condition of the left- and the right-moving sectors. Actually, the Dirichlet boundary condition

$$X = X_L + X_R = a \text{ (constant)} \tag{3.2}$$

relates the left- and the right-moving coordinates at the boundary and can be considered as a matching condition of X_L and X_R. Since the Dirichlet condition is mapped to the Neumann condition via the T-duality transformation $X_R \to -X_R$, we can also regard the Neumann condition as a matching condition of the two chiral sectors. To summarize, we have two types of matching conditions for a single coordinate X:

$$X_L = X_R + a \quad (\text{Neumann})\,, \tag{3.3a}$$

$$X_L = -X_R + a \quad (\text{Dirichlet})\,. \tag{3.3b}$$

From these matching conditions, one can obtain the gluing condition for the currents $J = i\partial X_L$ and $\tilde{J} = i\bar{\partial} X_R$.[7] To see this, suppose a holomorphic function $f(z)$ and an anti-holomorphic one $\bar{f}(\bar{z})$ satisfying the matching condition $f(z) = \bar{f}(\bar{z})$ along the boundary $z + \bar{z} = 0$.[8] Then, along the boundary, their derivatives are related as follows:

$$f'(z) = \frac{d}{dz} f(z) = \frac{d}{dz} \bar{f}(-z) = -\bar{f}'(\bar{z})\,. \tag{3.4}$$

Applying this relation to X_L and X_R, the matching condition (3.3a) of the Neumann type implies

$$J = i\partial X_L = (-i\bar{\partial})(X_R + a) = -i\bar{\partial} X_R = -\tilde{J}\,, \tag{3.5}$$

which is exactly the Neumann-type gluing condition.

We extend this argument to the case of group manifolds. In the WZW model on a group G, the G-valued field $g(z,\bar{z})$ satisfies the equation of motion $\partial(g^{-1}\bar{\partial} g) = 0$, which can be solved in the form

$$g(z,\bar{z}) = g_L(z) g_R(\bar{z})^{-1}\,. \tag{3.6}$$

In the same way as the case of S^1, we should consider that the fields g_L and g_R take values in the complexification $G^{\mathbb{C}}$ of G. If we take the Minkowski metric on the worldsheet, it is possible to decompose g within G. Note that this decomposition is determined up to a constant group element. In fact, g is invariant under the substitution $g_L \to g_L M$, $g_R \to g_R M$, $M \in G$. Using the above decomposition (3.6) of g in the expressions (2.14) for the currents, one obtains

$$J = -k\partial g_L\, g_L^{-1}\,, \qquad \tilde{J} = -k\bar{\partial} g_R\, g_R^{-1}\,. \tag{3.7}$$

As is explained in the previous section, the gluing condition (2.15) of the Neumann type corresponds to a D-brane on the conjugacy class. More precisely, the gluing condition restricts the form of the wave function to the class function. Since

[7] In this paper, we use the word "matching" for the boundary condition including the chiral zero modes, while the word "gluing" is used for the condition on the currents, i.e. without the zero modes.

[8] Note that we take the closed-string point of view, in which the boundary of the worldsheet is placed at $\mathrm{Re}\, z = 0$.

the gluing condition is linear on the wave function, we can take any superposition of the class functions as the wave function. Consequently, the position of D-brane is not completely fixed by the gluing condition. In order to specify the position of D-brane, we have to treat directly the boundary condition including the zero modes, i.e. the matching condition, instead of the gluing condition of the currents.

We have seen that the boundary condition in the case of S^1 can be rewritten as a matching condition of the left- and the right-moving sectors including the chiral zero modes. The boundary condition for the WZW model also admits the formulation as a matching condition. Our proposal is as follows:

$$g_L = g_R t\,, \quad t \in G\,, \quad \text{along the boundary } z+\bar{z}=0\,. \tag{3.8}$$

Because of the ambiguity in the decomposition (3.6) of g, t is determined up to a conjugation by a group element. In fact, t transforms as $t \to M^{-1}tM$, under the change of variables $g_L \to g_L M$, $g_R \to g_R M$, $M \in G$. Therefore, we can specify only the class $\mathcal{C}(t)$, not a single element t, in the matching condition (3.8). One can confirm that this condition reproduces the boundary condition $g|_{z+\bar{z}=0} \in \mathcal{C}(t)$ given in (2.17). Actually,

$$g = g_L g_R^{-1} = g_R t g_R^{-1} \in \mathcal{C}(t)\,. \tag{3.9}$$

The gluing condition (2.15) of the currents also follows from the matching condition (3.8):

$$J = -k\partial g_L\, g_L^{-1} = -k(-\bar{\partial})(g_R t)t^{-1}g_R^{-1} = k\bar{\partial}g_R\, g_R^{-1} = -\tilde{J}\,. \tag{3.10}$$

We can also write the twisted boundary condition $g|_{z+\bar{z}=0} \in \mathcal{C}^\omega(t)$ in the same way:

$$\omega(g_L) = g_R t\,, \quad t \in G\,, \quad \text{along the boundary } z+\bar{z}=0\,. \tag{3.11}$$

For the group element g, we obtain

$$g = g_L g_R^{-1} = g_L t\omega(g_L)^{-1} \in \mathcal{C}^\omega(t)\,. \tag{3.12}$$

The currents satisfy the following equation:

$$\tilde{J} = -k\bar{\partial}g_R\, g_R^{-1} = k\,\omega(\partial g_L\, g_L^{-1}) = \omega(-J)\,. \tag{3.13}$$

Since J can be written as $\sum_a J^a T^a = \sum_a J^{\omega(a)}\omega(T^a)$, the above equation implies

$$\tilde{J}^{\omega(a)} = -J^a\,, \tag{3.14}$$

which is exactly the twisted gluing condition (2.34).

3.2 Free field realization

We have seen that a Neumann-type boundary condition can be derived from the matching condition (3.8) of g_L and g_R. We use this fact to obtain the boundary states for the D-brane wrapped around the conjugacy class of SU(2).

In order to write the boundary states for the matching condition (3.11) explicitly, we adopt here the free field realization of $\mathrm{SL}(2,\mathbb{R})$ [15, 16, 17]

$$g = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{\gamma} & 1 \end{pmatrix} . \tag{3.15}$$

Note that $\gamma, \bar{\gamma}$ and ϕ are real fields for $g \in \mathrm{SL}(2,\mathbb{R})$. In terms of these fields, the worldsheet action S for the $\mathrm{SL}(2,\mathbb{R})$ WZW model with level $\tilde{k}$ can be written as

$$S = \frac{1}{4\pi} \int d^2 z \left(\partial\varphi\bar{\partial}\varphi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}e^{-2\varphi/\tilde{\alpha}_+} - \frac{2}{\tilde{\alpha}_+}\varphi\sqrt{g}R \right) , \tag{3.16}$$

where β and $\bar{\beta}$ are auxiliary fields conjugate to γ and $\bar{\gamma}$, respectively. φ is related with ϕ by

$$\varphi = \tilde{\alpha}_+ \phi , \quad \tilde{\alpha}_+ = \sqrt{2\tilde{k} - 4} . \tag{3.17}$$

For large ϕ, one can treat the screening charge perturbatively, and it is possible to consider (β, γ) and $(\bar{\beta}, \bar{\gamma})$ as free fields. We therefore have two pairs of bosonic first-order systems and a free boson:

$$\begin{aligned}
\gamma(z) &= \sum_n \gamma_n z^{-n} , \qquad \bar{\gamma}(\bar{z}) = \sum_n \bar{\gamma}_n \bar{z}^{-n} , \\
\beta(z) &= \sum_n \beta_n z^{-n-1} , \quad \bar{\beta}(\bar{z}) = \sum_n \bar{\beta}_n \bar{z}^{-n-1} , \\
\varphi(z, \bar{z}) &= \varphi_L(z) + \varphi_R(\bar{z}) , \\
\varphi_L(z) &= x_L - ip_L \ln z + i \sum_{n\neq 0} \frac{1}{n} \alpha_n z^{-n} , \\
\varphi_R(\bar{z}) &= x_R - ip_R \ln \bar{z} + i \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n \bar{z}^{-n} .
\end{aligned} \tag{3.18}$$

These fields satisfy the following OPE,

$$\begin{aligned}
\varphi(z, \bar{z})\varphi(w, \bar{w}) &\sim - \ln |z - w|^2 , \\
\beta(z)\gamma(w) \sim \frac{1}{z - w} , &\qquad \bar{\beta}(\bar{z})\bar{\gamma}(\bar{w}) \sim \frac{1}{\bar{z} - \bar{w}} .
\end{aligned} \tag{3.19}$$

The commutation relations among the modes are

$$[\beta_m, \gamma_n] = \delta_{m+n} , \qquad [x_L, p_L] = i , \qquad [\alpha_m, \alpha_n] = m\delta_{m+n} . \tag{3.20}$$

We regard SU(2) with level k as $\mathrm{SL}(2,\mathbb{R})$ with $\tilde{k}=-k$. We therefore obtain

$$\varphi = -i\alpha_+\phi\,, \quad \alpha_+ = i\tilde{\alpha}_+ = \sqrt{2k+4}\,, \tag{3.21}$$

for the case of SU(2) instead of (3.17).

One can express the currents J^a and $\tilde{J}^a$ as differential operators acting on the functions of $(\gamma,\bar{\gamma},\phi)$:

$$\begin{aligned}
J^+ &= -\frac{\partial}{\partial\gamma}\,, & \tilde{J}^+ &= -\bar{\gamma}^2\frac{\partial}{\partial\bar{\gamma}} + \bar{\gamma}\frac{\partial}{\partial\phi} + e^{-2\phi}\frac{\partial}{\partial\gamma}\,,\\
J^3 &= -\gamma\frac{\partial}{\partial\gamma} + \frac{1}{2}\frac{\partial}{\partial\phi}\,, & \tilde{J}^3 &= \bar{\gamma}\frac{\partial}{\partial\bar{\gamma}} - \frac{1}{2}\frac{\partial}{\partial\phi}\,,\\
J^- &= \gamma^2\frac{\partial}{\partial\gamma} - \gamma\frac{\partial}{\partial\phi} - e^{-2\phi}\frac{\partial}{\partial\bar{\gamma}}\,, & \tilde{J}^- &= \frac{\partial}{\partial\bar{\gamma}}\,.
\end{aligned} \tag{3.22}$$

In terms of the free fields, these currents can be written as follows

$$\begin{aligned}
J^+ &= -\beta\,, & \tilde{J}^+ &= -\bar{\beta}\bar{\gamma}^2 + \bar{\gamma}\alpha_+ i\bar{\partial}\varphi + k\bar{\partial}\bar{\gamma}\,,\\
J^3 &= -\beta\gamma + \frac{1}{2}\alpha_+ i\partial\varphi\,, & \tilde{J}^3 &= \bar{\beta}\bar{\gamma} - \frac{1}{2}\alpha_+ i\bar{\partial}\varphi\,,\\
J^- &= \beta\gamma^2 - \gamma\alpha_+ i\partial\varphi - k\partial\gamma\,, & \tilde{J}^- &= \bar{\beta}\,.
\end{aligned} \tag{3.23}$$

The Sugawara energy-momentum tensor takes the form

$$T = \frac{1}{k+2}J^aJ^a = \beta\partial\gamma - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{\alpha_+}i\partial^2\varphi\,. \tag{3.24}$$

The vertex operator $V_{j,m}$, which has the su(2) spin j and $J_0^3=m$, can be written as

$$V_{j,m} = \gamma^{j-m}e^{i\frac{2j}{\alpha_+}\varphi}\,. \tag{3.25}$$

This operator has the conformal dimension $\Delta_j = \frac{j(j+1)}{k+2}$. Since we have a non-vanishing 2-point function $\langle V_{-j-1,-m}(z)V_{j,m}(0)\rangle \neq 0$, the operator $V_{-j-1,-m}$ can be considered as the dual of $V_{j,m}$.

From the expression (3.15) for g, we can read off the left- and the right-moving part of g as follows:

$$\begin{aligned}
g_L &= \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-\phi_L} & 0 \\ 0 & e^{\phi_L} \end{pmatrix},\\
g_R &= \begin{pmatrix} 1 & 0 \\ -\bar{\gamma} & 1 \end{pmatrix}\begin{pmatrix} e^{\phi_R} & 0 \\ 0 & e^{-\phi_R} \end{pmatrix},
\end{aligned} \tag{3.26}$$

where ϕ_L and ϕ_R are the left- and the right-moving part of ϕ, respectively. The left-moving part g_L is an upper triangular matrix while g_R is a lower triangular one. In other words, g_L (g_R) is an element of the Borel group B_+ (B_-).[9] Since B_+

[9]Here, the Borel group B_+ is a group of all the upper triangular matrices in $\mathrm{SL}(2,\mathbb{C})$.

overlaps with B_- only at the diagonal matrices, the matching condition $g_L = g_R t$ is not suitable for the free field expression (3.26). In order to find an appropriate condition, we regard SU(2) as a subgroup of SL$(2,\mathbb{C})$. This is also necessary to perform an analytic continuation from SU(2) to SL$(2,\mathbb{R})$, for which the free field realization (3.15) is introduced. In SL$(2,\mathbb{C})$, we have a non-trivial involution $g \to (g^\dagger)^{-1}$, which reduces to a trivial one for SU(2). Correspondingly, we have two types of extensions of the condition $g_L = g_R t$ to SL$(2,\mathbb{C})$:

$$g_L = g_R t\,, \tag{3.27a}$$
$$(g_L^\dagger)^{-1} = g_R t\,. \tag{3.27b}$$

If we stay in SU(2), these two conditions are identical. However, $(g^\dagger)^{-1} \neq g$ for $g \in$ SL$(2,\mathbb{C})$ and the difference between these two expressions is meaningful. Here we take the latter choice (3.27b) to describe the D-brane in SU(2). Since the involution $g \to (g^\dagger)^{-1}$ maps B_+ to B_-, the condition (3.27b) can be applied to the free field expression (3.26).

As noticed above, we take the free fields $(\gamma, \bar{\gamma}, \phi)$ to be real. Hence, the matching condition (3.27b) can be written as follows:

$$\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} e^{\phi_L} & 0 \\ 0 & e^{-\phi_L} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\bar{\gamma} & 1 \end{pmatrix} \begin{pmatrix} e^{\phi_R + ia} & 0 \\ 0 & e^{-\phi_R - ia} \end{pmatrix}. \tag{3.28}$$

Here we write the element $t \in \mathbb{T}/W$, which represents the conjugacy class, as $t = \left(\begin{smallmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{smallmatrix}\right)$, $a \in [0, \pi]$. From this expression, the boundary condition for the free fields follows immediately

$$\gamma = \bar{\gamma}\,, \qquad \varphi_L = \varphi_R + \alpha_+ a\,, \tag{3.29}$$

where we used the relation (3.21) to rewrite ϕ with φ. One can see that φ satisfies the Neumann-type matching condition (3.3a). In terms of modes, this condition can be written as

$$\begin{aligned} &\gamma_n - \bar{\gamma}_{-n} = 0\,, \qquad && x_L - x_R = \alpha_+ a\,, \\ &\beta_n + \bar{\beta}_{-n} = 0\,, \qquad && \alpha_n + \tilde{\alpha}_{-n} = -\frac{2}{\alpha_+}\delta_{n,0}\,, \end{aligned} \tag{3.30}$$

where we denote p_L and p_R as α_0 and $\tilde{\alpha}_0$, respectively. The condition for β and $\bar{\beta}$ follows from that for γ and $\bar{\gamma}$.

The condition for α_0 and $\tilde{\alpha}_0$ differs from the usual Neumann-type. This is because, as is seen from (3.24), φ has a background charge. To understand this, let us consider a current $(W(z), \widetilde{W}(\bar{z}))$ with background charge Q, which has the following OPE with the energy-momentum tensor $T(z)$:

$$T(z)W(w) \sim \frac{1}{(z-w)^3}(-Q) + \frac{1}{(z-w)^2}W(w) + \frac{1}{z-w}\partial W(w)\,. \tag{3.31}$$

In terms of modes, we have

$$[L_m, W_n] = -nW_{m+n} - \frac{Q}{2}m(m+1)\delta_{m+n}\,. \tag{3.32}$$

Suppose that $W(z)$ and its anti-holomorphic counterpart $\widetilde{W}(\bar{z})$ satisfy the gluing condition

$$W_n + \widetilde{W}_{-n} = 0\,, \quad n \neq 0\,. \tag{3.33}$$

Assuming that the boundary condition preserves the conformal invariance, one can show

$$0 = [L_m - \tilde{L}_{-m}, W_{-m} + \widetilde{W}_m] = m(W_0 + \widetilde{W}_0 - Q)\,, \tag{3.34}$$

which means that the boundary condition consistent with the OPE (3.31) should be

$$W_n + \widetilde{W}_{-n} = Q\delta_{n,0}\,. \tag{3.35}$$

Substituting $(i\partial\varphi_L,\, i\bar{\partial}\varphi_R)$ for $(W, \widetilde{W})$, we obtain the boundary condition (3.30) for α_n and $\tilde{\alpha}_{-n}$.

The involution $g \to (g^\dagger)^{-1}$ acts on the element T of the horizontal Lie algebra as $T \to -T^\dagger$. Hence the current J^a is mapped as follows: $(J^+, J^3, J^-) \to (-J^-, -J^3, -J^+)$. Accordingly, from the matching condition (3.27b), we obtain the gluing condition

$$J^+_n - \tilde{J}^-_{-n} = 0\,, \qquad J^3_n - \tilde{J}^3_{-n} = 0\,, \qquad J^-_n - \tilde{J}^+_{-n} = 0\,. \tag{3.36}$$

Note that $(-\tilde{J}^-, -\tilde{J}^3, -\tilde{J}^+)$ have exactly the same form as (J^+, J^3, J^-) if we substitute $(\beta, \gamma, \varphi_L)$ for $(\bar{\beta}, \bar{\gamma}, \varphi_R)$ in (3.23). The above gluing condition therefore follows immediately from the boundary condition (3.30).

3.3 Boundary states

Let us now describe the construction of the boundary states satisfying the boundary condition (3.30). We treat (β, γ) and φ separately, and put them together after completion of the states in each sector.

We begin with the (β, γ)-sector. The boundary condition takes the form

$$\gamma_n - \bar{\gamma}_{-n} = 0\,, \qquad \beta_n + \bar{\beta}_{-n} = 0\,. \tag{3.37}$$

This condition preserves the ghost number current $(J_{\beta\gamma}, \tilde{J}_{\beta\gamma}) = (\beta\gamma, \bar{\beta}\bar{\gamma})$ except for the zero mode. Since the ghost number current has anomaly in the OPE with the energy-momentum tensor, we can apply the previous argument, in particular (3.35), to obtain the relation for the ghost numbers

$$Q_{\beta\gamma} + \widetilde{Q}_{\beta\gamma} = -1\,, \tag{3.38}$$

where $(Q_{\beta\gamma}, \widetilde{Q}_{\beta\gamma}) = (\oint J_{\beta\gamma}, \oint \tilde{J}_{\beta\gamma})$.

As is well known, the ground state of the (β, γ)-system is labeled by the bosonic sea-level, which is called a picture. More specifically, the vacuum $|q\rangle$ in the q-picture is characterized as follows:

$$\begin{aligned} \gamma_n \, |q\rangle &= 0\,, \quad n \geq 1+q\,, \\ \beta_n \, |q\rangle &= 0\,, \quad n \geq -q\,. \end{aligned} \tag{3.39}$$

Note that the vacuum $|q\rangle$ has non-vanishing ghost number, namely,

$$Q_{\beta\gamma} \, |q\rangle = q \, |q\rangle \ . \tag{3.40}$$

Hence, the picture of the left-moving sector is related with that of the right-moving sector through eq. (3.38), and the ground state takes the form $|q\rangle_L \otimes |-1-q\rangle_R$. On this ground state, we can construct the boundary state satisfying the condition (3.37) as follows:

$$|q\rangle\!\rangle_{\beta\gamma} = \exp\left[-\sum_{n \geq -q} \gamma_{-n} \bar{\beta}_{-n} - \sum_{n \geq 1+q} \beta_{-n} \bar{\gamma}_{-n} \right] |q\rangle_L \otimes |-1-q\rangle_R \ . \tag{3.41}$$

Let us consider the φ-sector. The boundary condition is

$$x_L - x_R = \alpha_+ a\,, \qquad \alpha_n + \tilde{\alpha}_{-n} = -\frac{2}{\alpha_+} \delta_{n,0}\,, \tag{3.42}$$

which is almost the Neumann condition except for the zero mode. We label the ground state of the oscillators α_n by the momentum $p_L = \alpha_0$

$$\alpha_n \, |k\rangle = 0\,, \quad n > 0\,, \qquad p_L \, |k\rangle = k \, |k\rangle \ . \tag{3.43}$$

It is convenient to introduce another parametrization of the states defined by

$$|n) = \left| \frac{n-1}{\alpha_+} \right\rangle . \tag{3.44}$$

The momentum n takes integer value since $|n)$ corresponds to the vertex operator $e^{i\frac{n-1}{\alpha_+}\varphi}$ with spin $\frac{n-1}{2}$, which should be half integer for the unitary irreducible representation. In this notation, the boundary condition $\alpha_0 + \tilde{\alpha}_0 = -\frac{2}{\alpha_+}$ is solved in the form $|n)_L \otimes |-n)_R$. One can see, from (3.25), that the su(2) spin j is related with the momentum n as $n = 2j+1$, and that $|-n)$ is dual of $|n)$. The boundary state is obtained by gluing the state with spin $j = \frac{n-1}{2}$ and its dual with spin $-j-1$. Since the condition for the oscillators is of the usual Neumann-type, we can write the boundary state $|n\rangle\!\rangle_\varphi$ in the φ-sector as follows:

$$|n\rangle\!\rangle_\varphi = \exp\left[-\sum_{m>0} \frac{1}{m} \alpha_{-m} \tilde{\alpha}_{-m} \right] |n)_L \otimes |-n)_R \,. \tag{3.45}$$

We treat the boundary condition for the zero mode later.

We combine the above results, (3.41) and (3.45), to obtain the full boundary state subject to the boundary condition (3.30), namely,

$$|n\rangle\rangle = \begin{cases} |n\rangle\rangle_\varphi \otimes |0\rangle\rangle_{\beta\gamma} & \text{if } n > 0\,, \\ |n\rangle\rangle_\varphi \otimes |-1\rangle\rangle_{\beta\gamma} & \text{if } n < 0\,. \end{cases} \tag{3.46}$$

Here we take the picture of the (β, γ)-system as $q = 0$ or -1 according to the sign of the momentum n. With this choice of the picture, the state with positive spin $j \geq 0$ always appears in the 0-picture while their dual with spin $-j-1$ does in the -1-picture. It is helpful to write down the first several terms of $|n\rangle\rangle$ and $|-n\rangle\rangle$:

$$\begin{aligned} |n\rangle\rangle &= |n)_L \otimes |-n)_R - \gamma_0|n)_L \otimes \bar{\beta}_0|-n)_R + \frac{1}{2}\gamma_0^2|n)_L \otimes \bar{\beta}_0^2|-n)_R + \cdots\,, \\ |-n\rangle\rangle &= |-n)_L \otimes |n)_R - \beta_0|-n)_L \otimes \bar{\gamma}_0|n)_R + \frac{1}{2}\beta_0^2|-n)_L \otimes \bar{\gamma}_0^2|n)_R + \cdots\,. \end{aligned} \tag{3.47}$$

From this expression, one can see that the left-moving sector of $|n\rangle\rangle$ is a highest-weight representation of su(2) while that of $|-n\rangle\rangle$ is a lowest-weight one,[10] since β_0 is the raising operator of su(2) (see eq. (3.23)). In other words, the choice of the picture (0 or -1) corresponds to the choice of the representation (highest-weight or lowest-weight).

Since the conjugacy class does not change by the action of the Weyl group W, the corresponding boundary state also should be invariant under the action of W. For su(2), $W = \mathbb{Z}_2$, and its non-trivial element acts on the weight λ as the reflection $\lambda \to -\lambda$. As we show above, the left-moving sector of the boundary state $|n\rangle\rangle$ is a highest-weight representation of su(2). Hence, the Weyl reflection of $|n\rangle\rangle$ has a lowest-weight representation in the left-moving sector. If we write $n = 2j+1$ for $n > 0$, the highest weight of $|n\rangle\rangle$ is j while the lowest-weight of $|-n\rangle\rangle$ is $-j$. Therefore, the Weyl reflection of $|n\rangle\rangle$ is $|-n\rangle\rangle$, and the boundary state $|n\rangle\rangle^W$ invariant under the Weyl group W takes the following form:

$$|n\rangle\rangle^W \equiv |n\rangle\rangle + |-n\rangle\rangle\,. \tag{3.48}$$

This state $|n\rangle\rangle^W$ should be considered as a building block of the boundary state for the D-brane on the conjugacy class.

We can write the states conjugate to $|n\rangle\rangle$ by using the hermitian conjugate of the modes:

$$\gamma_n^\dagger = \gamma_{-n}\,, \qquad \beta_n^\dagger = -\beta_{-n}\,, \qquad \alpha_n^\dagger = -\alpha_{-n}\,. \tag{3.49}$$

Here, γ is taken to be real. We take φ to be an anti-hermitian field since φ is related with the real boson ϕ through $\varphi = -i\alpha_+\phi$. Therefore the state $|n)$ has non-vanishing

[10] Note that the Fock space of the free fields forms a reducible representation of su(2). One needs to take the BRST cohomology in order to obtain an irreducible representation [17].

inner product with $|-n)$, which we normalize as $(-n|n) = 1$. Together with the (β,γ)-sector, we obtain

$$(-n, q=-1|n, q=0) = 1\,. \tag{3.50}$$

The explicit form of $\langle\!\langle -n| = |-n\rangle\!\rangle^\dagger$ for $n>0$ is

$$\begin{aligned}\langle\!\langle -n| &= {}_L(-n, q=-1| \otimes {}_R(n, q=0| \\ &\quad\times \exp\left[\sum_{m\geq 1}\gamma_m\bar{\beta}_m + \sum_{m\geq 0}\beta_m\bar{\gamma}_m - \sum_{m>0}\frac{1}{m}\alpha_m\tilde{\alpha}_m\right].\end{aligned} \tag{3.51}$$

We can calculate the annulus amplitude (2.25) using the boundary state (3.46) and its conjugate (3.51). The result is

$$\langle\!\langle -n|q^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{24})}z^{\frac{1}{2}(J_0^3+\tilde{J}_0^3)}|n\rangle\!\rangle = \begin{cases}\hat{\chi}_n^0(\tau,\nu) & \text{if } n>0\,,\\ \hat{\chi}_n^{-1}(\tau,\nu) = -\hat{\chi}_n^0(\tau,\nu) & \text{if } n<0\,.\end{cases} \tag{3.52}$$

Here we introduced the character $\hat{\chi}_n^q$ of the free field Fock space F_n^q, which is freely generated from the ground state $|n)\otimes|q\rangle$ by the action of the creation modes. For the 0-picture, which corresponds to the highest-weight representation, we obtain

$$\hat{\chi}_n^0(\tau,\nu) = \operatorname*{Tr}_{F_n^0} q^{L_0-\frac{c}{24}}z^{J_0^3} = \frac{q^{\frac{n^2}{4(k+2)}}z^{\frac{n}{2}}}{\vartheta_{1,2}(\tau,\nu) - \vartheta_{-1,2}(\tau,\nu)}\,. \tag{3.53}$$

One can also show that $\hat{\chi}_n^{-1}(\tau,\nu) = -\hat{\chi}_n^0(\tau,\nu)$. Hence, the lowest-weight representation (-1-picture) behaves in the character like the highest-weight representation (0-picture) with negative norm. For the W-invariant state $|n\rangle\!\rangle^W$, the annulus amplitude takes the form

$$\begin{aligned}{}^W\langle\!\langle n|q^{\frac{1}{2}(L_0+\tilde{L}_0-c/24)}z^{\frac{1}{2}(J_0^3+\tilde{J}_0^3)}|n\rangle\!\rangle^W &= \hat{\chi}_n^0(\tau,\nu) + \hat{\chi}_{-n}^{-1}(\tau,\nu)\\ &= \hat{\chi}_n^0(\tau,\nu) - \hat{\chi}_{-n}^0(\tau,\nu)\,.\end{aligned} \tag{3.54}$$

This amplitude can be regarded as a regularized inner product of $|n\rangle\!\rangle^W$ with itself. One can verify that, for each power of q, a finite number of states contribute to the inner product (3.54). This follows from the fact that the inner product (3.54) is written in the form of a difference of two characters $\hat{\chi}_n^0$ and $\hat{\chi}_{-n}^0$. Since $\hat{\chi}_n^0$ is the character of the representation with the highest-weight $\frac{n-1}{2}$, the number of states that contributes to the lowest power of q can be calculated as $\frac{n-1}{2} - \frac{-n-1}{2} = n$. This number expresses the norm of the wave function corresponding to the boundary state $|n\rangle\!\rangle^W$. The normalized boundary state is therefore $\frac{1}{\sqrt{n}}|n\rangle\!\rangle^W$.

In order to complete the boundary state for the conjugacy class, we have to take an appropriate linear combination of the states (3.48) to form a δ-function corresponding to the boundary condition $x_L - x_R = \alpha_+ a$ for the zero mode of φ.

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We should consider functions on $S^1/\mathbb{Z}_2$, since a parametrizes the position of the conjugacy classes that takes values in $\mathbb{T}/W \cong S^1/\mathbb{Z}_2$. From Weyl's integral formula, the relevant functions are the numerator of the SU(2) character (2.19): $\sin((2j+1)\theta)$, $j = 0, 1/2, 1, \ldots$. These functions are orthogonal to each other with respect to the following inner product:

$$\int_0^\pi d\theta \sin(m\theta)\sin(n\theta) = \frac{\pi}{2}\delta_{m,n}\,. \tag{3.55}$$

The δ-function for these functions can be written as

$$\delta(\theta, a) = \frac{2}{\pi}\sum_{n\geq 1}\sin(na)\sin(n\theta)\,. \tag{3.56}$$

Since the norm of $\sin(n\theta)$ is independent of n, we can identify the wave function $\sin(n\theta)$ with the normalized boundary state $\frac{1}{\sqrt{n}}|n\rangle\rangle^W$.

If we take this δ-function to describe the wave function of the zero modes, the boundary state for the D-brane wrapped around the conjugacy class at $\theta = a$ can be written as

$$|a\rangle\rangle_{\text{brane}} = \sum_{n\geq 1}\sin(na)\frac{1}{\sqrt{n}}|n\rangle\rangle^W\,. \tag{3.57}$$

This should be regarded as the boundary state valid in the $k \to \infty$ limit, since the annulus amplitude (3.54) differs from the $\widehat{\text{su}}(2)$ character $\hat{\chi}_j$ if k is finite. Actually, one can express $\hat{\chi}_j$ in terms of the free field character $\hat{\chi}^0_n$:

$$\hat{\chi}_j = \sum_{l\in\mathbb{Z}}\hat{\chi}^0_{2j+1+2(k+2)l} - \sum_{l\in\mathbb{Z}}\hat{\chi}^0_{-2j-1+2(k+2)l}\,. \tag{3.58}$$

Hence, we obtain $\hat{\chi}_j \sim \hat{\chi}^0_{2j+1} - \hat{\chi}^0_{-2j-1}$ in the limit of $k \to \infty$, which coincides with the annulus amplitude (3.54). Since this is the property that characterizes the Ishibashi state (see (2.22)), we can identify $|2j+1\rangle\rangle^W$ with the $k \to \infty$ limit of the Ishibashi state $|j\rangle\rangle_I$. After this identification, the boundary state (3.57) turns out to be the $k \to \infty$ limit of the Cardy's state, for which the wave function is the classical one (2.18). The boundary state (3.57), which realizes the matching condition (3.8), correctly reproduce Cardy's state in the limit $k \to \infty$.

In order to treat the case of finite k, we need to take into account the quantum effects in the construction of the boundary states. We can do this by imposing the invariance of the boundary states under the Weyl group $\hat{W}$ of $\widehat{\text{su}}(2)$ instead of W. The Weyl group $\hat{W}$ is a semi-direct product of the Weyl group W of su(2) and the translation of the weight lattice. Hence, the boundary state invariant under $\hat{W}$ takes the following form:

$$|n\rangle\rangle^{\hat{W}} = \sum_{l\in\mathbb{Z}}|n + 2(k+2)l\rangle\rangle + \sum_{l\in\mathbb{Z}}|-n + 2(k+2)l\rangle\rangle\,. \tag{3.59}$$

Here, we modify the assignment of the picture to be compatible with the action of the Weyl group

$$|n+2(k+2)l\rangle\!\rangle = \begin{cases} |n+2(k+2)l\rangle\!\rangle_\varphi \otimes |0\rangle\!\rangle_{\beta\gamma} & \text{if } 0<n<k+2\,, \\ |n+2(k+2)l\rangle\!\rangle_\varphi \otimes |-1\rangle\!\rangle_{\beta\gamma} & \text{if } -k-2<n<0\,. \end{cases} \tag{3.60}$$

Then the annulus amplitude for $|n\rangle\!\rangle^{\hat{W}}$ gives the $\widehat{\mathrm{su}}(2)$ character $\hat{\chi}_{\frac{n-1}{2}}$:

$$\begin{aligned} {}^{\hat{W}}\langle\!\langle n| q^{\frac{1}{2}(L_0+\tilde{L}_0-c/24)} z^{\frac{1}{2}(J_0^3+\tilde{J}_0^3)} |n\rangle\!\rangle^{\hat{W}} &= \sum_{l\in\mathbb{Z}} \hat{\chi}^0_{n+2(k+2)}(\tau,\nu) + \sum_{l\in\mathbb{Z}} \hat{\chi}^{-1}_{-n+2(k+2)}(\tau,\nu) \\ &= \sum_{l\in\mathbb{Z}} \hat{\chi}^0_{n+2(k+2)}(\tau,\nu) - \sum_{l\in\mathbb{Z}} \hat{\chi}^0_{-n+2(k+2)}(\tau,\nu) \\ &= \hat{\chi}_{\frac{n-1}{2}}(\tau,\nu)\,. \end{aligned} \tag{3.61}$$

Hence, we can identify the $\hat{W}$-invariant boundary state $|2j+1\rangle\!\rangle^{\hat{W}}$ with the Ishibashi state $|j\rangle\!\rangle_I$. Since $|n+2(k+2)\rangle\!\rangle^{\hat{W}} = |n\rangle\!\rangle^{\hat{W}}$, the normalization factor also should have the periodicity of $2(k+2)$. The most simple possibility is to take $1/\sqrt{[n]_q}$, $q = e^{i\frac{2\pi}{k+2}}$, instead of $1/\sqrt{n}$. If we adopt this factor, the quantum boundary state for the D-brane on the conjugacy class takes the following form:

$$|a\rangle\!\rangle_{\text{brane}} = \sum_{0<n<k+2} \sin(na)\frac{1}{\sqrt{[n]_q}} |n\rangle\!\rangle^{\hat{W}}\,. \tag{3.62}$$

Identifying $|n\rangle\!\rangle^{\hat{W}}$ with the Ishibashi state, the above state exactly coincides with Cardy's state (2.26).

4. Discussion

In this paper, we have constructed the boundary state for the spherical 2-brane wrapped around the conjugacy class of SU(2) directly in terms of the group variable. The resulting state coincides with Cardy's state. Since we have started from the geometrical setting that the worldvolume is a conjugacy class, this result clarifies the geometrical meaning of Cardy's state. Our construction is parallel to the one for the D-brane in flat background. Namely, we have decomposed the group variable into the left- and the right-moving coordinates on the worldsheet, and rewrite the boundary condition for a string ending on the conjugacy class of SU(2) as the matching condition of the chiral coordinates. This formulation of the problem is preferable to the ordinary boundary condition since it treats both of the chiral zero modes equally. In order to write the boundary state subject to the matching condition, we have used the free field realization of the WZW model, in which each chiral sector takes values in the Borel subgroup (B_+, B_-) of $\mathrm{SL}(2,\mathbb{C})$. The chiral field $g_L(z)$ ($g_R(\bar{z})$) is an upper (lower) triangular matrix. We cannot simply impose the matching condition to the left and right chiral sectors. Instead, we have used the involution $g \to (g^\dagger)^{-1}$, which is trivial in SU(2) but maps B_+ to B_-. With this involution, the matching condition for the group variables gives a simple boundary condition for the free fields.

As we have argued in this paper, the wave function $\widehat{\psi}^k_\alpha$ for Cardy's state takes the following form:

$$\widehat{\psi}^k_\alpha(\theta) = \frac{S_{\alpha 0}}{\sqrt{S_{00}}} \sum_{j=0,1/2,1,\ldots,k/2} \sqrt{\frac{2j+1}{[2j+1]_q}} \chi_j(\pi \frac{2\alpha+1}{k+2}) \overline{\chi}_j(\theta)\,. \tag{4.1}$$

In the limit $k \to \infty$, this wave function reduces to the δ-function concentrated on the conjugacy class:

$$\psi^k_\alpha(\theta) = \sum_{j=0,1/2,1,\ldots} \chi_j\left(\pi \frac{2\alpha}{k}\right) \overline{\chi_j(\theta)}\,. \tag{4.2}$$

For finite k, however, $\widehat{\psi}^k_\alpha$ is not a δ-function in the usual sense. Since the sum is restricted to the integrable representations, the peak of $\widehat{\psi}^k_\alpha$ is broadened and the worldvolume of the corresponding D-brane gets smeared. This is because the worldsheet theory is a rational CFT in which the number of primary fields is finite. Correspondingly, the Hilbert space of the wave function is finite dimensional, which suggests that the target space geometry differs from the ordinary one. In refs. [22, 23], the geometry behind a rational CFT is discussed in the framework of the non-commutative geometry. Since the worldvolume of the D-brane is a submanifold in the target space, we expect that the D-brane in a rational CFT also exhibits the feature of the non-commutative geometry. Actually, it is pointed out [5, 6] that, by calculating the algebra of functions on the worldvolume, the spherical brane in the SU(2) WZW model is a quantized sphere, which reduces to the fuzzy sphere [24, 25] in the $k \to \infty$ limit. Therefore, it is natural to expect that the above wave function $\widehat{\psi}^k_\alpha$ can be regarded as a generalization of the ordinary δ-function to the case of the non-commutative geometry. As we have discussed in the last section, the factor $[2j+1]_q$ in $\widehat{\psi}^k_\alpha$ is originated from the change of the normalization of the Ishibashi state $|j\rangle\!\rangle_I$. This change of the normalization suggests that the inner product of the wave function is q-deformed, which may also be interpreted in terms of the non-commutative geometry. One way to understand the relation between the brane and the non-commutative geometry is to compare the field theory on the quantized sphere [26, 27] with the effective field theory on the worldvolume [8].

Our boundary state is equipped with the structure of the free field resolution of the irreducible representation by imposing the invariance under the Weyl group of the current algebra. In particular, the Weyl reflection exchanges a highest-weight representation with a lowest-weight one, and we need both of them to construct the boundary state invariant under the Weyl reflection. In the calculation of the character, the lowest-weight representation behaves like the highest-weight representation with negative norm, and we reproduce the same structure as the character formula. Since our construction is geometrical, it may enable us to clarify the geometrical interpretation of the BRST cohomology in the free field realization of the current algebra.

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